AMS 517 Final Report
An extension of Davis and Lo's contagion model

Wai Chi Kan, Shanshan Li, Xiang Shi

May 8, 2013

1 Introduction

Contagion models were first introduced into the credit risk field by Davis and Lo in 2001. The idea behind contagion model is that default of a firm is either spontaneously or due to the default of other firms which we called it infected default. Davis and Lo's Model is a simple static model for estimating the probability distribution of total number of defaults in a market. The model uses identically independent Bernoulli random variables to describe the default probabilities. However, the Davis and Lo model is not enough to explain the dynamic cases that we have to due with loss over time in real practice. A more general extension model is therefore proposed by Cousin et al. in 2011 in which multi-period calculation is considered. The extension model also accounts for the domino effects of defaults which is not supported in the Davis and Lo's Model. Also, it introduces additional dependencies in the model and provides a more flexible contagion mechanism by allowing defaulting to occur under certain circumstance with different function settings. The authors calibrated the model with iTraxx tranche quotes to calibrate the model parameters. In this report, we briefly introduce the Davis and Lo's Model in Section 2 and the details of the extension model is presented in Section 3. The theoretical result and numerical application are discussed in Section 3 and 4 respectively.

2 Davis and Lo's model

Davis and Lo's model categorizes the cause of a bond's default into two ways: the bond may either **default directly** (spontaneous default) or may **be infected by any defaulting bond** (infectious default). Consider there are n bonds in the market or in a portfolio, there is a $\Omega = \{1, \ldots, n\}$ representing the set of indices of bonds. For a bond i, a Benoulli random variable X^i equals to 1 if the bond defaults directly or the bond is infectious default when the indicator random variable ζ^i equals to 1. Then we said the bond i is defaulted if $Z^i = 1$ and expresses in term of $X^i$and ζ^iin the following way:

$$Z^i = X^i + (1 - X^i)\zeta^i$$

The infection occurs when there is at least one of the other bond j defaults directly ($X^j = 1$,where $j \in \Omega, j \neq i$) and it infects the bond i ($Y^{ji} = 1$) as shown in the follows:

$$\zeta^i = 1 - \prod_{j \in \Omega, j \neq i} (1 - X^j Y^{ji})$$

Both $X^i$and $Y^{ji}$are i.i.d Bernoulli random variables, the model assumes the probability for bond i to default is p, so $P(X^i) = p$; and the probability for bond i being infected by bond j is q , $P(Y^{ji}) = q$ where $p, q \in [0, 1]$. From the model we can see that an infected bond cannot further infect another bond. The Davis and Lo's model is then stated that the distribution of the number of defaults in the market is given by,

$$P(\sum_{i \in \Omega} Z^i = k) = C_n^k \alpha_{nk}^{pq}$$

where

$$\begin{aligned}\alpha_{nk}^{pq} =& p^k(1-p)^{n-k}(1-q)^{k(n-k)} + \\ & \sum_{i=1}^{k-1} C_k^i p^i (1-p)^{n-i} (1-(1-q)^i)^{k-i} (1-q)^{i(n-k)}\end{aligned}$$

The former term in α is the probability of k bonds default directly and the latter terms is the probability that k bonds default in which i bonds directly default and k-i bonds are infectious default. Although the Davis and Lo's model is simple and easy to calculate, it does not explain the domino effect of default in the real world. Like during the financial crisis, the default of a firm may lead to a whole bunch of defaults of other firms and the infectious bonds can further infect the others that forms a chain effect and cause severe damage. So do the model static characteristic fails to provide cumulative defaults or losses over time that is essential to compute the CDO pricing.

3 The extension of the model

In this paper, the authors consider a time horizon $t \in \{1, \ldots, T\}$ and assume that the default can happen in each interval $(t-1, t]$. Suppose that there are n firms in the market which are indexed by $1, 2, \ldots, n$. And let Ω be the set of these firms. Then X_t^i is the indicator variable of direct defaultly, i.e. $\{X_t^i = 1\} = \{\text{firm i default directly at time t}\}$. Similar as Davis and Lo's model, ζ_t^i is denoted as the indicator of infection, i.e. $\{\zeta_t^i = 1\} = \{\text{firm i is infected at time t}\}$. Finally, let $\{Z_t^i, i \in \Omega\}$ which is a sequence of discrete random variables either equal to 1 or 0, denote the default process. Then the dynamics of Z_t^i can be obtained by the following recursive relation

$$\begin{aligned}Z_t^i &= Z_{t-1}^i + (1 - Z_{t-1}^i)[X_t^i + (1 - X_t^i)\zeta_t^i] \\ Z_1^i &= X_1^i + (1 - X_1^i)\zeta_1^i\end{aligned}$$

From this expression we can see that Z_t^i is a time series which are first zero, then jump to one when the firm i defaults. The default at time t is modeled by $X_t^i + (1 - X_t^i)\zeta_t^i$, which has the same structure as Davis and Lo's one period model.

This paper consider the following assumptions of the model:

Assumption 1 (Temporal independence of direct defaults). *$X_t = (X_t^1, \ldots, X_t^n)$, $t \in \{1, \ldots, T\}$ are mutually independent, but a dependency exists between the components of each vector.*

Assumption 2 (Temporal independence of exchangeable infections). *$Y_t = (Y_t^{11}, Y_t^{12} \ldots, Y_t^{nn})$, $t \in \{1, \ldots, T\}$ are mutually independent, and the random variables $\{Y_t^{ij}, (i, j) \in \Omega\}$ are exchangeable, independent of $\{X_t^i, t = 1, \ldots, T, i \in \Omega\}$.*

The second assumptions implies that all of the firms have the same chance to infect others. And note that the extension of the model is still based on the one-period structure because of the time-independence assumption. In the numerical and empirical tests of the paper, the author uses some stronger assumptions:

Assumption 3 (Temporal independence of exchangeable direct defaults)**.** $X_t = (X_t^1, \dots, X_t^n)$, $t \in \{1, \dots, T\}$ *are mutually independent, but the components of each vector are exchangeable random variables.*

In this case we will be allowed to apply De Finetti's Theorem to model the dependence structure of X_t and Y_t.

Assumption 4 (i.i.d. direct defaults and i.i.d. contaminations)**.** $\{X_t^i\}$ *and* $\{Y_t^{ij}\}$ *are mutually independent and Bernoulli distributed with parameter* p *and* q *respectively.*

This assumption is of Davis and Lo's model. The author would consider this in the numerical test for comparison.
In addition, the author introduce the following notations:

- Θ_t: the set of firms declared in default up to time t.
- Γ_t: the set of the firms which did not default in the previous t periods.
- F_t^i: the set of the infectious defaulting firms susceptible to infect firm i.
- N_t^D: the number of spontaneous defaults without external influence occurred during period t.
- N_t: the cumulated number of defaults occurred up to time t.

Note that the firms in the set Ω / F_t^i are the ones which will not affect firm i even they are defaulted. If firm j is included in F_t^i, then we can specify a random variable Y_t^{ji} as the indicator of contagion. $Y_t^{ji} = 1$ implies that the contamination link from an infectious firm j to a firm i is activated. Unlike Davis and Lo's model, the author assumes that the relationship between ζ_t^i and Y_t^{ji} has a more general form:

$$\zeta_t^i = f(\sum_{j \in F_t^i} Y_t^{ji}), \; i \in \Omega$$

where $f : \{0, \dots, n\} \to \{0, 1\}$. For example, setting $f(x) = 1_{x \geq 1}$ implies that any activated infection links causes an indirect default (Davis and Lo's model). In some cases the firm i will not default until it is infected by more than one firms, therefore we can specify $f(x) = 1_{x \geq 2}$ implies that two or more infections are necessary to generate a new default.

One of the important assumptions of the paper is that $Y_t^{ji}, j = 1, 2, \dots$ are exchangeable. Thus if the numbers of elements in $F_t^i, i = 1, \dots, n$, say $card(F_t^i)$ are the same, then so are the distributions of $\sum_{j \in F_t^i} Y_t^{ji}$. In this paper, the author assumes that $card(F_t^i)$ is given by

$$card(F_t) = g(N_{t-1}, N_t^D)$$

where $g : \mathbb{N}^2 \to \mathbb{N}$. By specifying g we can determine the different sources of contagion, for example:

- Inter-periodic contagion: $g(N_{t-1}, N_t^D) = N_{t-1}$. Only defaults up to time $t-1$ can infect other firms at time t.

- Intra-periodic contagion: $g(N_{t-1}, N_t^D) = N_t^D$. Only spontaneous defaults in the same period can infect others. Note that this is case of Davis and Lo's model.
- Contagion from external sources: $g(N_{t-1}, N_t^D) = n_0$. An arbitrary number n_0 denotes the sources outside our portfolio or bond space Ω which might be able to contaminate the bonds inside the portfolio.
- Combination of the three latter contamination modes. For instance, $g(N_{t-1}, N_t^D) = n_0 + N_t^D$ implies both external sources and spontaneous defaults in period t can affect other firms.

4 Theoretical results

The goal of this paper is to find the distribution of the cumulative default process N_t. Like Davis and Lo's model, the distribution function can be explicitly computed. Suppose that we know the distribution of X_t and Y_t, then we can define the following coefficients:

Definition 1 (Coefficient of order k)**.** *The coefficient of order k for the set $\{X_t^i, i \in \Gamma\}$ is*

$$\mu_{k,t}(\Gamma) = \frac{1}{C^k_{card(\Gamma)}} \sum_{j_1<j_2<\cdots<j_k} P(X_t^{j_1} = 1 \cap \cdots \cap X_t^{j_k} = 1)$$
$$\mu_{0,t}(\Gamma) = 1 \text{ (including if } \Gamma = \emptyset)$$

The coefficient of order k for the set $\{Y_t^{ij}, (i,j) \in \Gamma\}$ is

$$\lambda_{k,t} = P(Y_t^1 = 1 \cap \cdots \cap Y_t^k = 1), k \geq 1$$
$$\lambda_{0,t} = 1$$

The coefficient of order k for the set $\{\zeta_t^i, i \in \Gamma\}$ is

$$\xi_{k,t}(g(u,l)) = P(\zeta_t^1 = 1 \cap \cdots \cap \zeta_t^k = 1 | N_{t-1} = u, N_t^D = l), k \geq 1$$
$$\xi_{0,t}(g(u,l)) = 1$$

Note that if the assumption 3 holds, then $P(X_t^{j_1} = 1 \cap \cdots \cap X_t^{j_k} = 1)$ are equal for all different $j_1, \ldots, j_k$ because X_t are exchangeable. Therefore we would have a simpler expression of $\mu_{k,t} = P(X_t^1 = 1 \cap \cdots \cap X_t^k = 1)$. And that is the reason for which we can also set $\lambda_{k,t}$ to have a similar expression.

Since ζ_t^i is given by the function of Y_t^{ji}, the coefficient of order k for ζ_t^i is determined by $\lambda_{k,t}$. In fact, for any function f such that $\zeta_t^i = f(\sum_{j \in F_t^i} Y_t^{ji})$, we have

$$\xi_{k,t} = \sum_{\gamma=0}^{zk} \eta_{k,z}(\gamma) \sum_{j=0}^{zk-\gamma} C^j_{zk-\gamma}(-1)^j \lambda_{j+\gamma,t}$$

where $\eta_{k,z}(\gamma) = \sum_{\gamma_1 \in \{0,\ldots,z\}} f(\gamma_1) C_z^{\gamma_1} \eta_{k-1,z}(\gamma - \gamma_1)$, $\eta_{1,z}(x) = 1_{x \leq z} f(x) C_z^x$ and $\eta_{0,z}(x) = 1_{x=0}$.

Suppose that the distribution of X_t and Y_t are given, then we can obtain the main results of this paper. For the one period model

$$Z^i = X^i + (1 - X^i)\zeta^i, i \in \Omega$$

we have the following theorem:

Theorem 1. *under assumptions 1 and 2, if $T = 1$ then the default's number law is given by*

$$P(N = r) = P(\sum_{i\in\Omega} Z_i = r)$$
$$= C_n^r \sum_{k=0}^{r} C_r^k \sum_{\alpha=0}^{n-r} C_{n-r}^{\alpha} \xi_{\alpha+r-k,1}(g(0,k)) \sum_{j=0}^{n-k} C_{n-k}^{j} (-1)^j \mu_{j+k}(\Omega)$$

For the multi-period model

$$Z_t^i = Z_{t-1}^i + (1 - Z_{t-1}^i)[X_t^i + (1 - X_t^i)\zeta_t^i]$$

we have the following theorem:

Theorem 2. *under assumptions 1 and 2, the defaults' number law is given by*

$$P(N_t = r) = \sum_{\substack{\theta_t \subset \Omega \\ card(\theta_t)=r}} P(\Theta_t = \theta_t)$$

where Θ_t is the set of firms declared in default up to t and where

$$P(\Theta_t = \theta_t) = \sum_{u=0}^{r} \sum_{\substack{\theta_t \subset \Omega \\ card(\theta_t)=r}} P(\Theta_t = \theta_t | \Theta_{t-1} = \theta_{t-1}) P(\Theta_{t-1} = \theta_{t-1})$$

where

$$P(\Theta_t = \theta_t | \Theta_{t-1} = \theta_{t-1})$$
$$= \sum_{m=0}^{r-u} \sum_{\substack{M_t \subset \theta_t - \theta_{t-1} \\ card(M_t)=m}} \rho(M_t, \Omega - \theta_{t-1} - M_t) \sum_{j=0}^{n-r} C_{n-r}^{j} (-1)^j \xi_{j+r-u-m,t}(u,m)$$
$$\rho(A,B) = P(\forall i \in A, X_t^i = 1 \ \ et \ \forall i \in B, X_t^i = 0) \ \forall A, B \subset \Gamma_{t-1}$$

with $card(\theta_t) = r$ and $card(\theta_{t-1}) = u, u \leq r$.

Note that we would have a simpler expression of these two theorems if the assumptions 3 or 4 holds. So the only thing left is to model the distribution of X_t and Y_t. We know that the marginal distributions of X_t^i and Y_t^{ji} are Bernoulli random variables. So we need to specify the dependence structure of them. Suppose that the assumption 3 holds, that is, $X_t^1, \ldots, X_t^k$ are part of an infinite sequence of exchangeable Bernoulli random variables. Then by De Finetti's theorem, there exists a random variable Θ_{X_t} which takes value in $[0,1]$ and its cumulative distribution function $F_{\Theta_{X_t}}$ such that

$$P(X_t^1 = 1, \ldots, X_t^k = 1) = \int_0^1 \theta^k dF_{\Theta_{X_t}}(\theta)$$

Therefore, by specifying the distribution of Θ_{X_t}, the coefficient of order k for X_t is equal to the k-th moment of Θ_{X_t}:

$$\mu_{k,t} = E[\Theta_{X_t}^k]$$

Similarly, we can define the dependence structure of Y_t and have

$$\lambda_{k,t} = E[\Theta_{Y_t}^k]$$

In the rest of this paper, the author assume that for all t, Θ_{X_t} and Θ_{Y_t} follow the Beta distribution with mean p, q and variance σ_X, σ_Y respectively.

5 Numerical applications

In this part, we first investigate the effect of exchangeability, between direct defaults and infections. We then study the fit of the model to price synthetic CDO tranches. Using the certain restriction of the model , we could compare the calibration performance of the model on iTraxx data before and during the crisis.

5.1 Effect of model parameters on the dynamics of loss distribution

First, we define 4 reference models, which have some common characteristics: consider 10 firms, on a 10-period time interval, with a direct default probability $p = 0.1$ and infection probability $q = 0.2$. The four models are different in the feature of direct defaults or infections, i.i.d. or not and on the infections number required to cause a default. The four models are presented as follows:

- model 1: $\sigma_X = 0, \sigma_Y = 0, f(x) = \mathbf{1}_{x\geq 1}$ (i.i.d. case, one required contamination)
- model 2: $\sigma_X = 0, \sigma_Y = 0, f(x) = \mathbf{1}_{x\geq 2}$ (i.i.d. case, two required contamination)
- model 3: $\sigma_X = 0.2, \sigma_Y = 0.2, f(x) = \mathbf{1}_{x\geq 1}$ (exchangeable case, one required contamination)
- model 4: $\sigma_X = 0.2, \sigma_Y = 0.2, f(x) = \mathbf{1}_{x\geq 2}$ (exchangeable case, two required contamination)

Then, recall the function of $\xi_{k,t}(z)$, which gives the probability of loss distribution. We could get the the evolutions of expectation and variance of N_t as a function of t, for all of the four models.

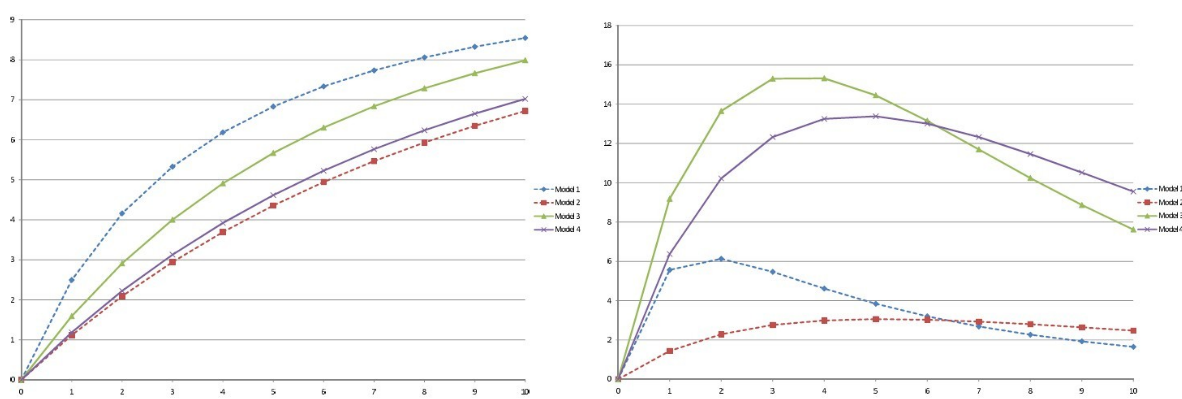

Figure 1: Evolution of $E[N_t]$ and $V[N_t]$ as a function of t

From the right side of Figure 1, the mean of the default number is increasing with time, which is intuitively consistent with the fact that we expect more defaults in a larger period of time. For models 3 and 4, the direct defaults and infections increase due to the hidden factors. For the group of models 1 and 2, the distribution of N_t is more dispersed, then the impact of such a dispersion is greater than that of models 3 and 4. For models 2 and 4, the contagion effect is weakened, so the mean of the loss distribution is mainly explained by the mean of the direct defaults. And that is the reason why their means are smaller.
From the left side of Figure 1, we can see that the variance of the loss distribution is a hump-shaped function of time. It is obvious that the dispersion level increases as time goes by. However, when the expected number of defaults comes to a critical threshold, the number of expected surviving firms decreases, then lead to a decrease in the dispersion level of the loss distribution. For models 1 and 3, increasment occurs earlier because the expected number of defaults is greater. For models 3 and 4, the variation in the hidden random variables implies a larger increase of the variance of N_t.

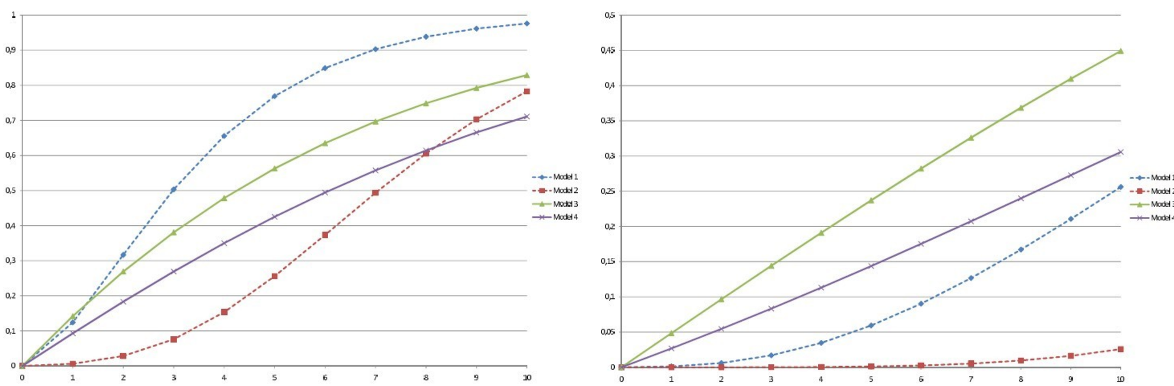

Figure 2: Evolution of $P(N_t \geq 6)$ and $P(N_t = 10)$ as a function of t

Figure 2 shows the evolution of $P(N_t \geq k)$, for respective fixed number $k = 6$ and $k = 10$. We can see that the growth of the variation of N_t does not lead to the increase of $P(N_t \geq 6)$, especially when this last probability is larger in the independence case. However, if we consider less frequent events that all firms default is plotted, we find that the impact of exchangeability is much more explicit. This case makes sense, since the increase of the dispersion level of N_t tends to emphasize the tail of the loss distribution.

5.2 Calibration on liquid CDO tranche quotes

Next, we will move on to examine the fit of the model to tranche spread of the 5 years iTraxx Europe main index, which is the most liquid segment part of the market, at two fixed points in time, March 1^{st} 2007 and January 31^{st} 2008, which are respectively before and during the credit crisis. In iTraxx Europe main index, we have 125 investment grade CDS. For each tranche, we define the attachment point as the level of subordination and the detachment point as the maximum loss of the portfolio. To price the standard tranches, we give the definition of the cumulative loss per unit of nominal exposure:

$$L_t = \sum_{i=1}^{n}(1 - R_i)Z_t^i$$

where R denotes the recovery rate associated with name i.

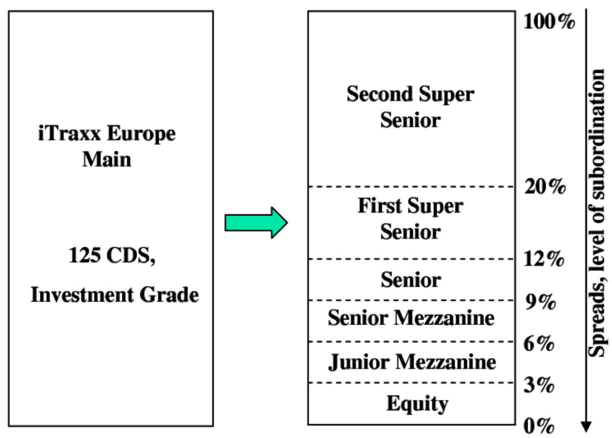

Figure 3:

Then, let us get back to the model we consider for the calibration of CDO tranche quotes. Here we use the beta-mixture model, the direct and contagion defaults are Bernoulli random variables while the common hidden parameter is Beta-distributed. And the infectious entities is given by $g(k, \gamma) = n_0 + \gamma$ where $n_0 = 1$. We know that the computation of CDO tranche spreads merely involves expectation of tranche losses. Because of the constant R, we can remark that the cumulative loss is just proportional to the number of defaults. So we can use N_t, $0 \leq t \leq T$, to compute CDO tranche spreads.

Furthermore, the contagion model can be described by the vector of parameters $\alpha = (p, \sigma_X, q, R)$. Our goal is to find out the optimal parameter set which minimizes the root mean square error:

$$RMSE(\alpha) = \sqrt{\frac{1}{6}\sum_{i=1}^{6}(\frac{\tilde{s}_i - s_i(\alpha)}{\tilde{s}_i})^2}$$

where $\tilde{s}_i$ denotes the market spreads and $s_i(\alpha)$ denotes the contagion model.

	0%-3%	3%-6%	6%-9%	9%-12%	12%-20%	index	RMSE
Market spreads 31 Jan 2008	31	317	212	140	74	77	-
Model spreads	32	328	204	142	77	64	7.5
Market spreads 1st Mar 2007	10	46	13	6	2	23	-
Model spreads	10	37	14	6	2	21	9.2

Table 1: iTraxx Europe market and model spreads (in bp) and the corresponding root mean square errors. The [0%-3%] spread is quoted in %. All maturities are for five years.

	p^*	σ_X^*	q^*	R^*
31 Jan 2008	0.0012	0.0151	0.0007	0.1964
1st Mar 2007	0.0001	0.0026	0.0005	0.1346

Table 2: Optimal parameters $\alpha^* = (p^*, \sigma_X^*, q^*, R^*)$ in a quarter period

From Table 1, the joint calibration of all tranche and the index is acceptable for 2008 crisis data and 2007 data, excepted for the $3\% - 6\%$ mezzanine tranche. As for the optimal parameters, we can see the great shift in credit spreads between these two periods. Even if the parameters are relatively small, they may trigger a significant expected number of defaults due to the large number of entities in the portfolio.

6 Conclusion

In this report, we study the extension contagion model proposed by Cousin et al. The extension model modifies the Davis and Lo's model from the original static form to a multi-period form. Meanwhile it relaxes some assumptions in the original model to make it more flexible and practical in real practice. By introducing multi-period recursive formula, the defaulted firms and infected firms in the previous periods can infect the firms in latter periods which allows the explanation of domino defaults. Specific functions can be set up to determine how many infections may lead to a default. Additional dependencies of random variables are proposed, the default distribution function can be depended on common macroeconomic factors. The authors has also investigated the use of mixture distributions (Beta and Bernoulli) for default related random variables. They validated the model by examining how the model parameters influence the loss distribution. After all, they used the iTraxx Europe Main Index data to calibrate the model parameters and the results fit well.